Using geometry to analyze the mass distribution of hand-held weapons

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1 Introduction

I have described in a previous article[1] how to measure the dynamic properties of swords or other hand-held weapons. Of particular significance in this endeavour are points called *centers of oscillation*[2]. Each point of the weapon has an associated center of oscillation, and there is a simple mathematical relationship between their positions that allows to find all pairs once you have measured just one.

The goal of this article is to expose a geometrical diagram that allows for the visual representation of this relationship, and more. That diagram can be drawn with straight edge and compass, though a set square will be a practical help for some parts.

Thanks to this diagram, I hope readers will be able to get a better intuitive grasp of how these points are related, and how they are modified when mass is added at any point on the object.

[1] http://www.subcaelo.net/ensis/weighing/weighing.pdf

[2] Most in the community concerned by the reproduction of swords know these points as 'pivot points'. I prefer to use 'center of oscillation' as it is the correct term for these in physics; another is 'center of percussion' but usage in the sword community has confused the meaning of the latter.

2 Basic construction of the diagram

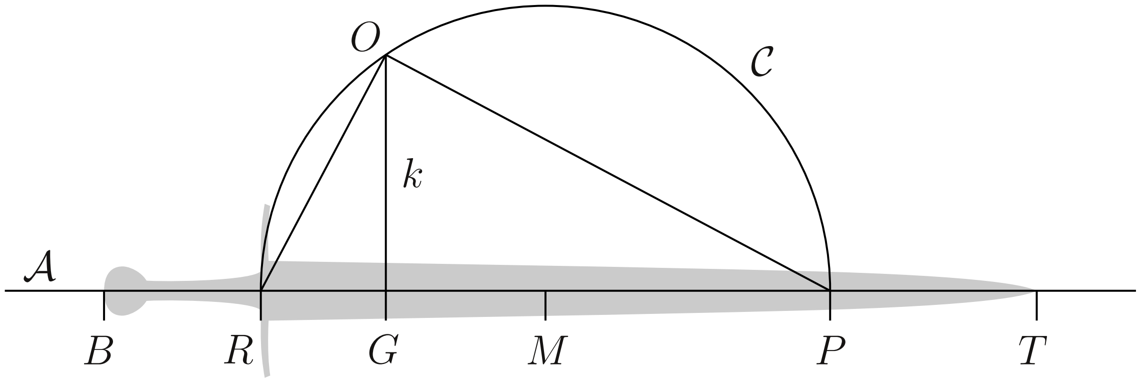

Figure 1

I will describe the construction step by step. Please refer to figure 1 for an illustration.

First draw a straight horizontal line $\mathcal{A}$ representing the longitudinal axis of the weapon. On this line, place point G, the center of gravity. This will form the baseline of the diagram. On this axis you can of course also place all the points and lengths needed, starting for example with B and T (the butt and tip of the weapon, respectively). To better materialize that axis I've drawn a shaded sword on the background, but this is purely cosmetic.

Suppose you have determined a pair of associated Centers of Oscillation R and P. Place them on the axis, and draw a semicircle $\mathcal{C}$ with diameter RP (its center is simply the midpoint M of the segment RP). Then, raise in G the perpendicular to the axis $\mathcal{A}$. This intersects $\mathcal{C}$ at point O. This point is central on the diagram and its position accurately represents the mass distribution of the weapon. I call this point the *representative* of the mass distribution. The vertical distance OG, by that construction, is the radius of gyration k, i.e. how far on average mass is located from the center of gravity. If that distance is small, mass is concentrated around G, conversely if it is big, it indicates that mass is spread over the length of the weapon, or even concentrated at the extremities.

Note that by that construction, the triangle ROP is a right triangle. This graphically represents the relationship between pairs of centers of oscillation and the radius of gyration. That relationship is of course true for any pair, and this will be used in the next section. See appendix A.1 for a mathematical proof of that property. The circle here is just used to build a right triangle.

3 Geometrical method for finding other centers of oscillation

Now that you have determined the origin O of the diagram, it is easy to find other pairs without having to measure anything on the weapon again.

If you have a reference point R' and want to find the corresponding center of oscillation P', all you have to do is to draw the segment $R'O$, then a line perpendicular to this segment in O. That line intersects the axis of the weapon in P', the center of oscillation associated to R'. As with the original points R and P, $R'OP'$ must be a right triangle.

That simple geometric construction, basically building a right triangle, makes it easy to find associated pairs and visualize how the centers of oscillation 'move' depending on the reference point, as illustrated on figure 2. If you have a set square you can set its right angle at O and rotate it around to see the associated centers of oscillation.

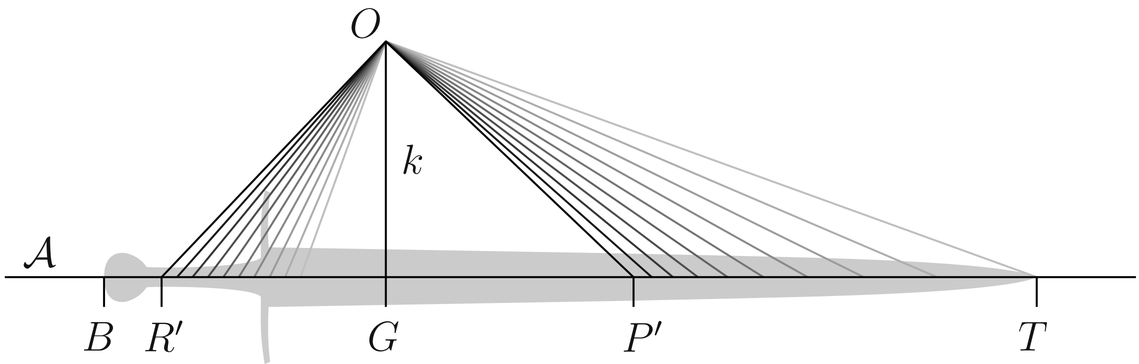

Figure 2

4 Representing modifications of the mass distribution

Another advantage of that geometrical construction is that it makes it possible to visualize the effects of modifications of mass distribution. I will detail here the example of adding mass at one point, but this can be generalized to any combination of two mass distributions. See figure 3.

Let us investigate what happens when a mass m_W is added to the weapon at point W (roughly the center of the pommel). It is easy to demonstrate (see appendix A.2) that the center of oscillation associated to W, V, is not changed. This will be the basis of the construction. The center of gravity of the new system is G'. That change of position depends on the relative value of m_W to the mass of the weapon m as per this formula:

$$WG' = \frac{m}{m + m_W} WG$$

Now we have a center of oscillation and the center of gravity of the new system, so it is easy to repeat the construction in section 2 and find the representative point O' for the new system, which unlocks all the centers of oscillation. For example we can find the new center of oscillation associated to R, P'.

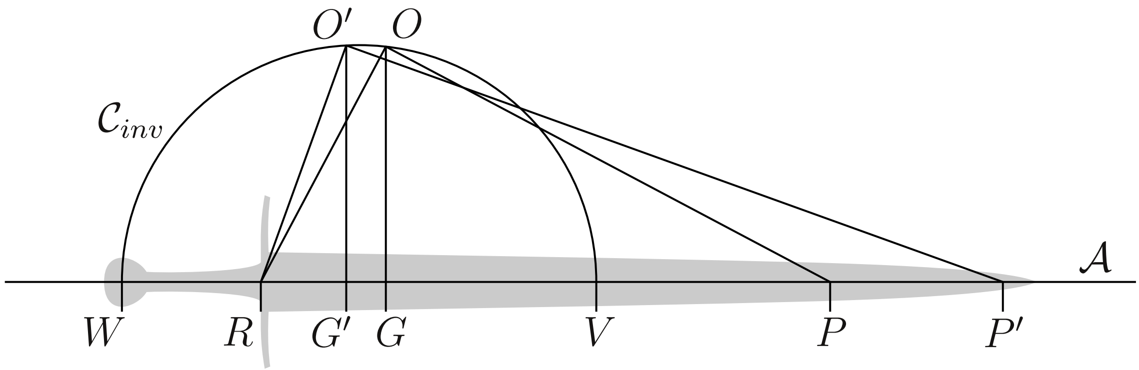

Figure 3

Note that by that construction, whatever the mass m_W is, the representative point of the final system always ends up on circle $\mathcal{C}_{inv}$, which has WV as its diameter and is invariant. This geometrically shows why adding mass to the pommel is not always a good way to alter the balance of a sword: only a few possibilities are open, point O moves along a fixed circle. Of course the appropriate position for O is not always on that circle, and you need to act on at least two points to be able to move it anywhere else.

Another use of this geometrical construction is to get an idea of where to work on the object in order to achieve a given mass distribution. If you know which representative point O' you want to achieve, and the current representative point O of the weapon, you just have to draw a circle with the center on the axis of the weapon going through O and O', and the intersection with the axis gives you the point where you should add mass.

5 Conclusion

All of the effects detailed here can be determined quicker and more accurately with the help of a modern computing device. However, being able to visualize and occasionally operate without such device has been tremendously helpful to me. It makes all these non-linear relationships intuitive on some level, which opens the gates to a qualitative understanding, at the very least.

Though the constructions exposed were all doable by medieval artists, I have not come across any direct proof that this has ever been the case. Theoretical considerations about centers of oscillation go back to the XVIIth century at least, but as far as I'm aware were not paired with such a geometrical interpretation. Therefore, I'm exposing this as a purely modern construct, useful to analyze artifacts of the past.

As I expose further properties of mass distribution in future articles, I will refer back to these geometrical constructions... Stay tuned!

6 Acknowledgments

I am grateful to Peter Johnsson for giving me the impulse to write this article. Though the research I expose here dates back to 2006-2009, his recent finds on the use of geometry for sword design [1] resonated with these other geometrical considerations, and made this article more relevant.

For those wondering, the data I have used here for the examples is actually measured on a reproduction by the French sword smith Gaël Fabre[3]. The sword is meant to be used for sword and buckler fight in the I.33-style. I wish to thank Gaël for the opportunity to measure, and for the good discussions we had about swords!

References

[1] Peter Johnsson. "One Single Wholeness of Things": The Geometry of Medieval Swords in the Wallace Collection. In *The Noble Art of the Sword: Fashion and Fencing in Renaissance Europe 1520-1630*, pages 142–149. Paul Holberton Publishing, 2012.

[3] `http://gael-fabre-forgeron.over-blog.com/`

A Demonstrations

A.1 Geometrical relationship between centers of oscillation

The reader should refer to figure 1.

Let's choose G as the origin of the coordinate system. Then the coordinates of R are $(r, 0)$, those of P are $(p, 0)$. Let us consider point O of coordinates $(0, k)$, the radius of gyration of the weapon. Let us demonstrate that if P is the center of oscillation associated to R , then the triangle ORP is a right triangle.

By applying the Pythagorean theorem we can find that $RO^2 = k^2 + r^2$ and $PO^2 = k^2 + p^2$. Since P is the center of oscillation associated to R we also have the physical relationship $-rp = k^2$. Then

$$\begin{aligned} RP^2 &= (p-r)^2 \\ &= r^2 + p^2 - 2rp \\ &= r^2 + p^2 + 2k^2 \\ &= (r^2 + k^2) + (p^2 + k^2) \\ &= RO^2 + PO^2 \end{aligned}$$

By the Pythagorean theorem again, ORP is a right triangle.

Using $\mathcal{C}$ to find point O is merely a way to build a right triangle, as any triangle which has three vertex on a circle and one side as the diameter of the circle is a right triangle (a form of Thales' theorem).

A.2 Invariance by addition of mass

The reader should refer to figure 3.

Let us suppose that we add mass m_W at W on the weapon. We have two things to demonstrate. First, the formula for the position of the center of gravity of the new system G'. Second, that point V remains the center of oscillation associated to W no matter the mass added.

By definition of the center of gravity, for any point X we have:

$$(m + m_W)\, XG' = m\, XG + m_W\, XW$$

Let us apply that relationship to point W. We find, since $WW = 0$:

$$(m + m_W)\, WG' = m\, WG$$

and finally

$$WG' = \frac{m}{m + m_W} WG$$

For the second point we will have to use a formula for the moment of inertia of the weapon around W, J_W. This is given by the parallel axis theorem:

$$J_W = m\,(k^2 + WG^2)$$

and knowing that V is the center of oscillation for W:

$$\begin{aligned} J_W &= m\,(WG \cdot GV + WG^2) \\ J_W &= m\,WG \cdot (WG + GV) \\ J_W &= m\,WG \cdot WV \end{aligned}$$

The same computation can be carried on the new system weapon + mass, and assuming that V' is the new center of oscillation for W in that new system, this gives:

$$J'_W = (m + m_W)\,WG' \cdot WV'$$

Now, the parallel axis theorem also says that by adding mass in W, we do not change the moment of inertia around W, because that mass would be at the center of rotation. Hence $J_W = J'_W$. And we can also use the relation we have found for WG'. We end up with:

$$m\,WG \cdot WV = m\,WG \cdot WV'$$

And this leaves us with $V = V'$ (except if $W = G$, then neither V nor V' are well defined, they are both at infinity).