A dynamic method for weighing swords

Vincent Le Chevalier *

November 15, 2014

1 Introduction

The main goal of this article is to provide a practical way to measure the mass distribution of swords, or at least the aspects of it that have a significant effect on handling properties. These meaningful measurements should ease the comparisons between weapons when hands-on examination is impractical. The typical situations for use of such characteristics could range from on-line reviews and purchase of reproductions or martial training tools, to the study of museum pieces.

Numerical properties are interesting in that they allow comparison even when measured by people from completely different backgrounds. In that, they are better than just appreciation of how the weapon handles with words. Words are always relative to what the writer has previously experienced. Numbers on the other hand, are just relative to each other.

Some familiarity with basic Newtonian physics will be useful in order to fully understand the reasoning, however this is not necessary for carrying out the measurements. For the more scientifically inclined individuals, several mathematical demonstrations are included as appendices.

Some may be worried by the idea of reducing the dynamic feel of things like swords, which feel immensely complex when you become familiar with them, to just a few numbers... The key point to underline here is that the complexity, trade-off, etc. is still present in the interaction and dependency between those numerical properties. Do not underestimate the possible complexity of the interaction between a handful of numbers...

This paper is not intended to provide a solution to the difficult problem of the modern, accurate reproduction of ancient artifacts. It is obvious that there is more to it than just a few numbers, and that nothing will ever replace hands-on examination. This is also caused by the fact that a sword, for example, has more qualities than just balance: it must be solid, sharp, resilient... Balance is just one of the facets we can look at when appreciating a sword, or any such weapon.

I must stress from the beginning that the aim is not to find a weapon *better* than the other. This simply has no meaning. The qualities of a weapon may be best suited to a fighting style or another, but most weapons can be used effectively (that is, from the point of view of mass distribution; some may still be unsuitable for fighting because of materials or assembly for example). It will be possible to check if a reproduction has the same properties as the original it is based on, but no characteristic should be interpreted as ‘it is best if the value is so much’. It should always be ‘it amounts to so much, thus it favours X at the expense of Y’.

And finally a word of caution: measuring a weapon involves some handling. Be careful when you do it with sharp blades. In fact be careful in general! I do not want people to get hurt while just trying to measure swords. If you feel your grip is not secure enough, if there are any living beings nearby (including yourself obviously) that could be hurt by a sword falling down, just lay the sword down and rest. If the sword falls and is not worth any piece of your own skin then do not attempt to catch it. Do not take chances...

1.1 Simplifying assumptions

For an easy analysis and measurement, we need to make some simplifying assumptions. For the purpose of analyzing mass distribution and its consequences on the dynamics of the weapons, we will consider the sword as a rigid, one-dimensional rod with a varying mass density along its length. This assumption is justified by three main observations:

- Swords are generally rigid, at least in the plane of the edge
- Swords are thin: their length is several times bigger than either their thickness or width
- Many swords are straight, or with a small degree of curvature

Of course this is an approximation, but analyzing simple objects such as these is a prerequisite for anything more complex. Vibrations cannot really be considered without including rigid body motions. Curved objects are harder to measure and their handling is particular, but some of the conclusions about straight objects are still useful. It is common and well-accepted for studies in physics to make those kinds of simplifying assumptions, and it leaves us with a model that is sufficiently complex for a first study.

2 Swords as two-mass systems

2.1 Insight from physics

Our simplified sword can still be quite complex. In theory, in order to fully describe it, one would need to specify the density of mass at each point along the length. That makes for a significant (technically infinite) amount of data, and it is quite impossible to fully measure without literally cutting the sword in very small pieces.

Fortunately, we can further simplify the description. Because we are interested in the consequences of mass distribution on handling and motion, we can look into Newton's equation of dynamics in order to judge which properties really are important. I will not spell out the equations here (see details in appendix B.1), but they show that only three intrinsic properties of the sword should be considered. These are:

- the total mass M
- the location of the center of gravity G
- the radius of gyration around the center of gravity k

The total mass is the amount of matter there is in the sword. The center of gravity is the average position of the matter. The radius of gyration is a length that represents how far on average the matter is from the center of gravity. Or, for those more familiar with statistics:

- mass $\approx$ number of samples
- center of gravity $\approx$ average
- radius of gyration $\approx$ standard deviation

This insight allows us to say that many of our one-dimensional rods are equivalent to one another: if two objects share the same M, G and k, and are subject to the same external actions at the same places, they move exactly in the same way. This means that they cannot be distinguished by any measurement procedure that only involves forces, torques and motions. They also feel the same because they react to our actions in the exact same way.

In that sense, there is some useless complexity even in the simplistic one-dimensional rod model. Swords are actually even simpler than that as far as their *perceived* mass distribution is concerned. What remains to do at this stage is to find a useful family of simple objects whose behaviour is intuitive, which can be linked to any sword, and whose defining properties can be measured. This is the topic of the next section.

2.2 Two-mass equivalent

A very useful and intuitive equivalent object is a *two-mass system*. It is simply built from two masses of different values, linked together by a rigid but mass-less rod. The advantage is that instead of having a multitude of density values along the whole length, we can consider only two masses and their respective positions. That makes four degree of freedom only.

Readers might have noted that the equations of dynamics show that only three parameters are required to fully determine an object's mass distribution. That means that for any given sword, you can build as many two-mass equivalents as you want, depending on only one parameter. For example, the location of one of the masses can be arbitrary. Figure 1 shows a pair of two-mass systems equivalent to a simple uniform stick.

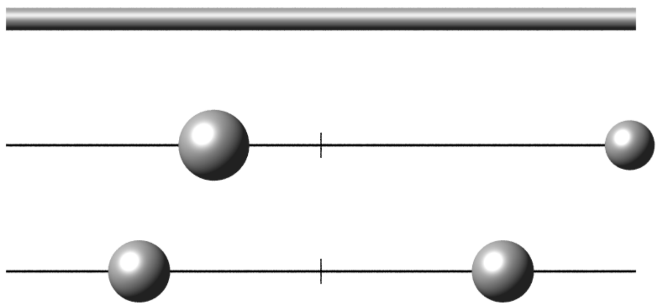

Figure 1: Several objects equivalent dynamic-wise to a simple stick, that would handle and move in the same way. From top to bottom: the stick with uniform density, a two mass system with one of the masses near the tip, another two-mass system with the masses equally spaced around the center of balance. The volume of the spheres here is proportional to their mass, but for the equivalence to be exact they would have to be point-masses.

Of course, once the location of one of the masses has been decided, it is possible to compute the location of the other one, as well as the values of both, from M, G and k. The converse is also true; from a system of two masses it is fairly easy to compute the three core variables. Mathematical demonstrations of these aspects can be found in appendix B.2. However these computations are not strictly necessary for a first approach, because as will be shown in section 3 it is possible to measure the properties of two-mass systems directly from the sword.

2.3 Analysis of the system applied to handling properties

Handling can be assessed if one of the masses is placed on the grip (remember, the position of one of the masses can be arbitrarily chosen in order to obtain a two-mass system). Let's call H that position on the grip. Then the other mass must be placed at another point F that lies on the blade (as has been pointed out before the exact place where the mass is can be measured, see section 3).

Once this is done there are three essential properties that can be distinguished:

- **dynamic length:** the length HF between the two masses.
- **blade weight:** the value of the mass located on the blade at F.
- **cross weight:** the value of the mass located at H.

Dynamic length and blade weight are the two key limiting factors for the performance of the sword. Dynamic length indicates how long the sword feels. Imagine handling a sword with the eyes closed: you will still have some feeling of length, of

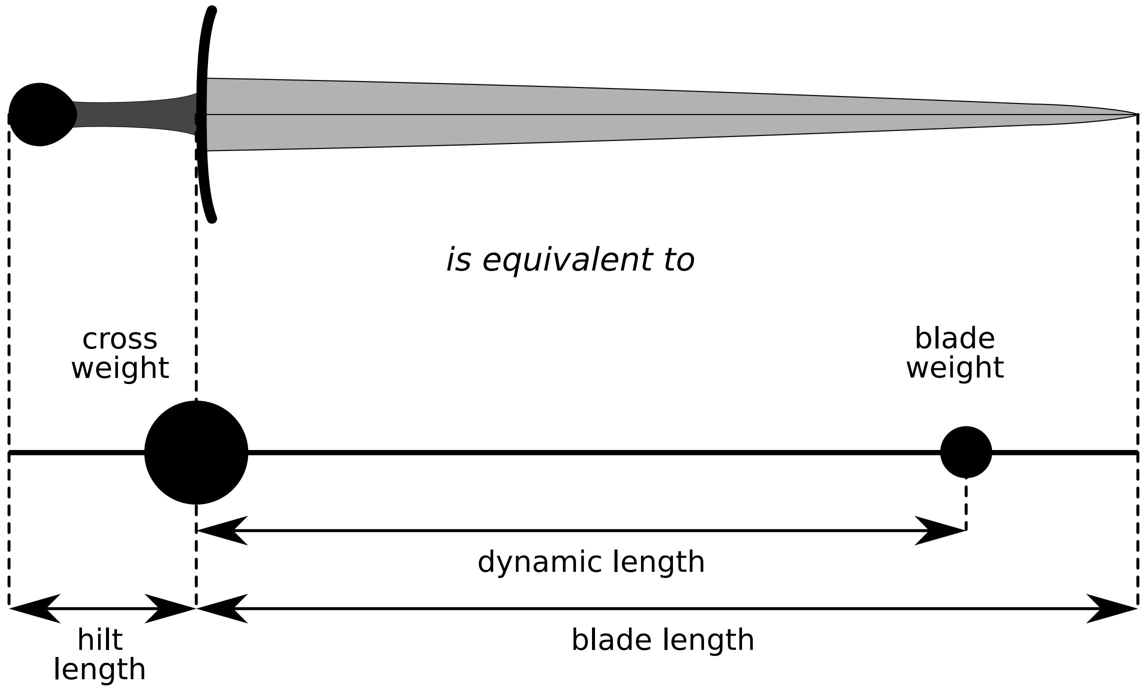

Figure 2: A sword and the dynamically equivalent two-mass system that can be used to assess handling. The drawing shows the five defining properties: blade length, hilt length, dynamic length, cross weight and blade weight.

where and how far the sword points, because you feel the dynamic length. Also, the longer it is, the more stable the sword will feel in thrusts, the shorter it is, the quicker it will snap into cuts. Blade weight is specifically felt in the wrist. Cross weight is less important for instantaneous perception in my experience but has a long-term effect : a big cross weight will tire the arm sooner. Of course the sum of blade weight and cross weight is the total mass of the sword... Blade weight and cross weight as defined here should not be understood as the weight of individual components of the sword, that can only be measured once the sword is disassembled. Rather, they are virtual weights that summarize the dynamic properties of the sword. For example, adding mass to the pommel will diminish blade weight, whereas the mass of the actual blade would obviously remain the same.

Blade weight is generally a good approximation of the mass that impacts the target when the sword hits. The two masses are not strongly coupled upon a sideways impact: when there is an impact at the location of one of them the motion of the other one is not changed and its mass does not come into account. Of course mass is not the only thing that matters for an impact, but it has effects, in particular a bigger mass will make deeper damages. On the other hand, a big mass is not really efficient if the target is light and mobile; in this case the target gets moved around a lot but is not irreversibly damaged.

What most people call ‘blade presence’ is actually fairly well represented by the ratio between the blade weight and the total weight. Note that it could be possible to have swords with the same blade weight, but with different blade presence. That is how some swords can have a significant blade presence, without necessarily being wrist-breakers : they have 'normal' blade weights but a relatively low cross weight.

An interesting issue is deciding on which exact point should be chosen as the point H. In my experience it is best to chose the junction between cross and handle as that reference. The main reasons for this is that this point is an objective reference that can be accurately located on most swords, and according to which we generally position our lead, dominant hand. It is therefore the natural reference point to chose when trying to compare the handling of swords.

Empirically, the dynamic length seems to be always very close to that of a uniform stick of the same length as that of sword. That means that a sword cannot have any random balance and still feel good. That being said, more data is needed to fully qualify this observation.

In addition to these mass distribution properties, two geometric lengths have a notable importance: blade length and hilt length. Blade length is the distance between the junction cross-handle and the tip, and defines the reach of the sword. Hilt length is measured from the junction cross-handle to pommel nut, and is somewhat less significant. It only indicates if the sword can be handled with one or two hands, and the maximum lever that can be used to move the sword if it is used with two hands.

All the properties are summed up on figure 2, and numerical results that I have gathered on several swords can be found in appendix A.

3 Measuring

A good thing about these two-mass systems is that all their properties can be directly measured in most pragmatic cases. This section describes the process.

3.1 General idea

Point H is easy to spot, which leaves us with the problem of finding point F (or equivalently measuring the dynamic length) and determining the masses.

Let us assume that we have a way to determine the dynamic length. Then it is trivial to measure both masses: one just has to rest the sword horizontally on two scales, at point H on one side and at point F on the other side, and read the values. That procedure can be further simplified and involve only one scale, the other being replaced by a simple resting point. That way, one of the masses is measured, and it is easy to obtain the total weight of the sword, hence the value of the other mass.

3.2 Measuring the dynamic length: using centers of oscillation

The dynamic length is a tricky parameter because it cannot be measured statically. You have to move the object in order to measure it, there are no ways around that. I will describe here the test that needs the least amount of equipment and can be performed by hand and eye.

The basic idea is easily understandable, once you know that the weapon is dynamically equivalent to a two-mass system. If you shake one of the masses very quickly, perpendicular to the rod, the other will stay nearly motionless. Taking gravity into account, this remains true only if the object is vertical, i.e. one mass above the other. The dynamic length is simply the distance between the 'shaking' location and the point that does not move somewhere on the sword.

Let us now describe the measurement from the practical point of view.

The sword must be grabbed as lightly and accurately as possible at the reference point H. I usually grab the weapon between just index and thumb. The goal here is to minimize the torque that can be applied to the weapon, because it perturbs the measurement. Let the weapon hang down vertically, with the center of gravity below your hand. It is usually easier and safer that way. The test can be done the other way around, but I do not recommend that. The sword is unstable when its center of gravity is above the holding point, this is distracting and dangerous when trying to get an accurate measurement.

Move that reference point back and forth as fast as possible. The motion must be quick but need not be too ample, around one fifth of the length of the weapon is plenty enough. If all goes well, you should see a point down on the sword that does not move. This is known as the *center of oscillation* associated to your reference point. It is also the point F where the second mass of the two-mass system lies. That process is illustrated on figure 3.

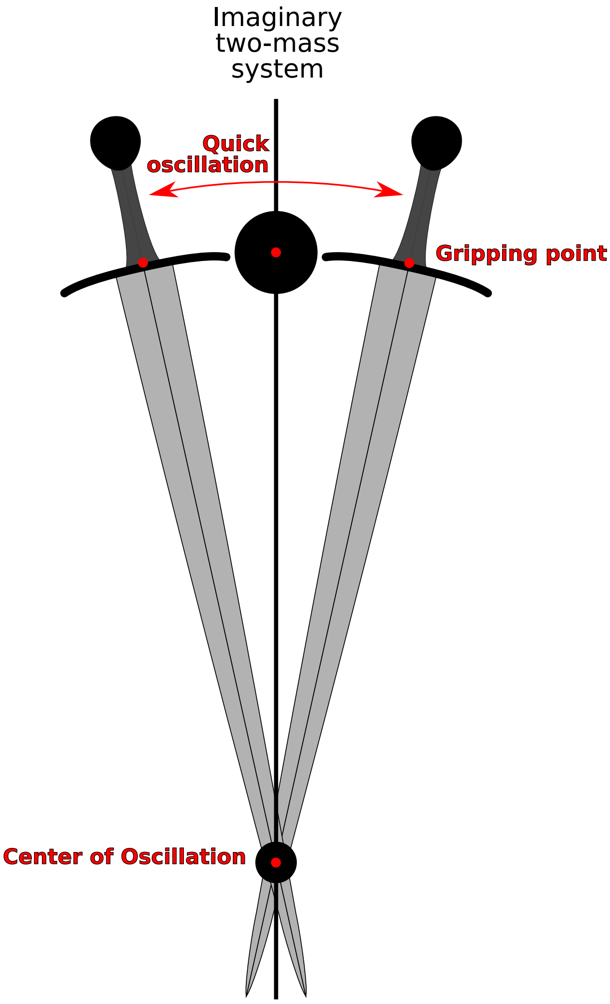

Figure 3: The so-called 'waggle test'.

There are several mistakes that can disturb this measurement. First, if you apply a torque on your gripping point, it will unsettle the oscillation. That is why the grip must be as light as possible. Second, if the motion of your hand is not quick enough, there will be a motionless point on the weapon, but it will not be the actual center of oscillation. In fact, by varying the frequency of your hand's motion, you should be able to make the weapon pivot about anywhere you want. It is only when the frequency becomes high that the motionless point merges with the center of oscillation.

These factors mean that this kind of measurement needs a little practice. Trying it with simple sticks at first is a good idea. It allows to get a feel of it with something safer and lighter than a sharp sword. I would expect competent martial artists to perform this test better, because they will be more comfortable with letting the weapon move by itself.

As far as precision is concerned, the main source of error lies in spotting the fixed point. It is really difficult to get it within less than a couple of centimeters. Looking at the sword from the top, handle to tip, makes it appear more clearly. As I said earlier, the frequency of the motion also has an influence. However, the error due to the frequency becomes small rapidly as frequency goes up, smaller than the error in spotting the fixed point. This is probably a bit of an academic concern though: the final error of the measurement might be smaller than what you can feel when handling swords... The governing equations are detailed in appendix B.3.

A good practical mean to mark and check the position of the center of oscillation is to use a hair tie looped around the blade, and look for its motion. If it is too close to H, it will move in phase with your hand, going left when you hand goes left. If it is too far, it will move in opposition, going right when your hand goes left. Through trial and error you can manage to get the hair tie just at the fixed point, and this makes it easier to measure the dynamic length.

3.3 Illustrated example

In this section I will give a step-by-step illustrated description of how to measure a sword for its handling properties. The sword under study here is a blunt rapier made by Darkwood Armory.

Let's start with the necessary tools:

- A tape ruler, at least as long as the sword you want to measure
- Some sort of scale to weigh your sword. I use a kitchen scale with a flat plate, able to measure up to 5kg and accurate to a gram, which is plenty sufficient
- One coloured hair tie, not too large for precision
- Some supporting objects that will probably have to be used in order to weigh the sword

This whole set can be seen on figure 4.

The first step is measuring the important lengths on the sword. I prefer to store at least blade length (cross-grip junction to tip) and overall length, or equivalently blade length and hilt length (overall length minus blade length). The most important to get precisely is the blade length. In order to do that, it is advisable to put the zero of the ruler at the tip, and then measure accurately the blade length and overall length from there. This process is illustrated on figure 5. Of course other significant lengths could be noted, for example grip, ricasso, etc.

As a second step, the dynamic length should be measured. The waggle test is used here, holding the sword at the handle-cross junction. The hair tie is used to get an easier and more accurate measurement. See the result also on figure 5, and a video of

Figure 4: The tools used for the measure of the dynamic properties of swords: scales, tape ruler, hair ties, supporting objects. And of course the sword...

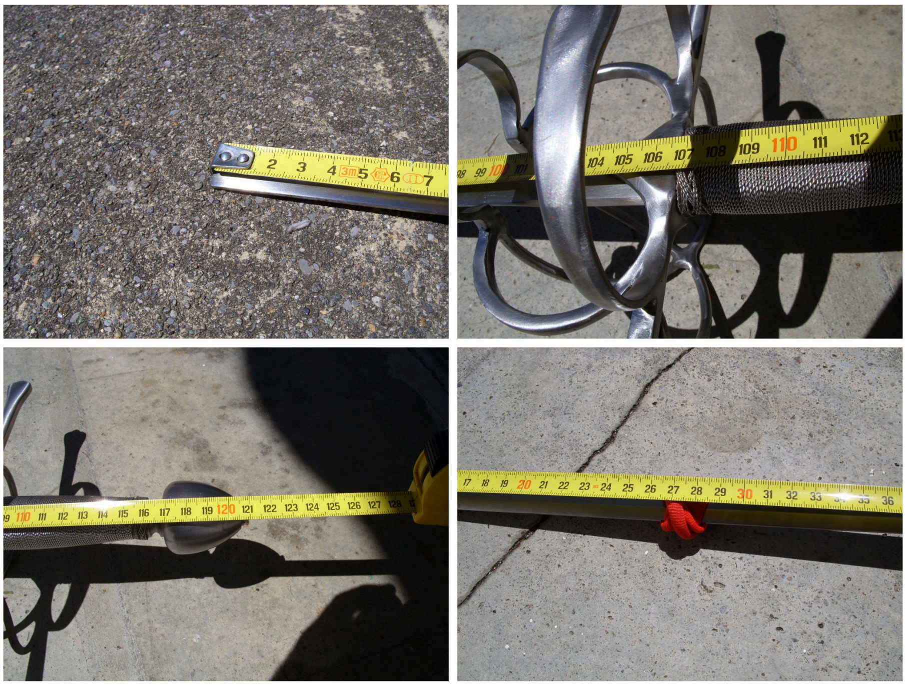

Figure 5: All the important lengths measured on a sword: putting the origin of the ruler at the tip (upper left), first get the blade length (upper right), the overall length (lower left) and the dynamic length (lower right, this involves a subtraction from blade length: here 107cm - 27.5cm = 79.5cm).

the waggle test on my website [1]. Leave the hair tie on, it will be useful in the next step.

Finally blade weight and total weight can be measured. It is possible that some kind of supporting object will be needed on the scales to be able to lay the sword as it should. Digital kitchen scales commonly have a function to take tare weight into account, do not forget to use it!

In order to measure blade weight, a small supporting object must be found so that the blade can rest on it as accurately as possible. A certain brand of chocolate candy has a shape that lends itself to that very well... The other resting point can be one of your fingers, at the cross-handle junction, as illustrated on figure 6. It is also important that the sword is as close to the horizontal as possible, otherwise errors might be introduced.

Measuring the total weight is so simple and well known that I do not feel compelled to provide an illustration...

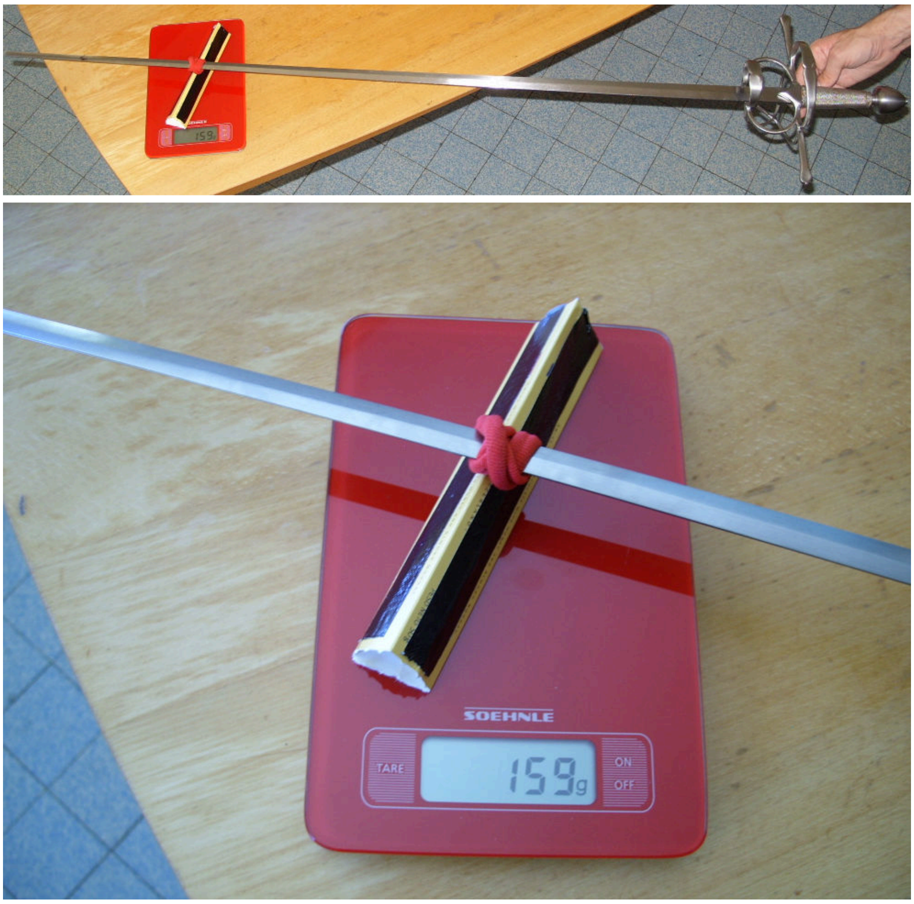

Figure 6: Weighing the blade mass. The sword rests horizontally on the finger at the handle-cross junction and on the support and scales at the center of oscillation, marked by the hair tie.

[1] `http://www.subcaelo.net/ensis/weighing/waggleTest.avi` or `http://www.subcaelo.net/ensis/weighing/waggleTest.wmv`

4 Conclusion

I hope this article can provide some more accurate understanding on how mass distributions of swords can be measured, interpreted and compared. Focusing on the quantities that have an effect on the handling properties of the weapons allows to reduce the number of necessary measurements. This kind of measurement should prove useful for any objective comparison between weapons.

What is needed now is more data, collected both on original weapons and reproductions; it would allow us to refine our understanding of how these significant quantities vary across sword types. That is why I have chosen to address the measuring process in greater details in my first article.

I have decided not to write down everything I know about mass distribution in this first article, focusing only on the most simple form of measurement that gives usable results without any computation. Aspects that I would like to discuss in further articles include:

- More advanced measurement procedures, using correlation between several centers of Oscillation
- How to obtain variations of the mass distribution when mass is added or removed somewhere (for example how to balance something)
- Different equivalent objects
- Various graphical representations of the results
- More in-depth interpretation of the handling properties
- Consequences on impact behaviour

All of these will require more supportive data, and more mathematical manipulations.

5 Acknowledgments

My work on these aspects has been really kicked off by George Turner's article Sword Motions and Impacts[2]. I disagree with some of the conclusions, but it is still a very good read, and the first article as far as I know to introduce 'pivot points' (that I refer to now as 'centers of oscillation') in a sword context. This has been a significant step forward.

I wish to thank the owners, administrators and moderators of the various web forums that facilitated many good discussions about sword balance, in particular MyArmoury[3] and Sword Forum International [4].

[2] Hosted by the ARMA: `http://www.thearma.org/spotlight/GTA/motions_and_impacts.htm`, or see the more complete version: `http://armor.typepad.com/bastardsword/sword_dynamics.pdf`

[3] `http://www.myarmoury.com`

[4] `http://forums.swordforum.com`

Sword makers were kind enough to discuss these matters publicly there. I am grateful to Angus Trim, Craig Johnson, Peter Johnsson and Michael 'Tinker' Pearce for all their practical observations and theoretical questioning, as well as for the design and production of some of the swords that were my experimental basis... I wish to give special thanks to Gaël Fabre, an impressive French swordsmith who kindly let me measure some of his swords at the Dijon events in 2009 and 2010.

Finally, I do not forget all the people I have discussed with over the years, mainly on forums but also by email, with special mention to Kyle Horn, Thom Ryan, Jean Thibodeau and David E. Farrell who also took the time to measure some of their swords and share the results. This has greatly contributed to my own understanding. The data provided by Thom Ryan is notable in quantity, quality and relevance and is included with his permission in the appendices of this article... And of course a word for my family and friends who tolerated my unusual passion with an impressive patience!

A Handling data measured on actual swords

I have gathered some data on my collection of swords, which has contributed a lot to my current understanding. Here is a short description of the swords:

A&A Milanese rapier The standard one from Arms&Armor offerings, bought in 2004;

Darkwood rapier A rapier I bought in 2007, with a blunt bated rapier blade, measured here with a 3 ring swept hilt. More details about that rapier can be read here: `http://www.myarmoury.com/talk/viewtopic.php?t=9865`;

Boken 1 A rather classic wooden Japanese training weapon made out of white oak. Quite lively and even-balanced;

Boken 2 Another wooden weapon, heavier and more difficult to handle, that I bought in order to increase my speed and power;

Iaito A metal training Japanese saber. This particular example is very lively, almost too light;

Ninja to A typical, a-historical straight and short blade you see in movies. This is a cheap reproduction that I bought when I was too young to know better;

Albion Squire The standard offering from Albion Armourers Next Gen line;

AT Type XI An Angus Trim sword, actually the first functional steel sword I have bought, back in 2001;

Longsword waster This is a longsword made by Purpleheart Armoury, in 2003 I believe (lost track of the exact date);

Napoleonic briquet A small marine infantry weapon (it is the model 'An XI' to be accurate), this seems to be an original;

G.F. XV-1/2 Two swords made by Gaël Fabre, a French swordsmith, of very good quality. Both are of Oakenshott Type XV;

G.F. XVI-1/2 Two other swords by Gaël Fabre, this time of Oakenshott Type XVI;

G.F. Spatha Another sword by Gaël Fabre, very different from the other four. It is a Merovingian Spatha sporting a wide fullered blade with parallel edges, far more cut-oriented than the rest;

A&A Montante A steel training sword produced by Arms&Armor meant to be representative of the larger Iberian two-handers;

Synthetic A one-handed synthetic sword produced by the Knight Shop, designed as a relatively safe and cheap sparring and drilling tool. Some leaders of Historical European Martial Arts based in Europe took part in the design of this model. This is the version used at the HEMAC event Dijon 2010; newer versions might have different properties...

A fellow sword enthusiast, Thom Ryan, also posted some data he measured on swords of his collection. I am grateful to him for the time he took and decided to include the data deduced from the measurements here. The list of his swords is as follow:

A&A Cavalier The standard offering from Arms&Armor, described as allowing a very steady and powerful thrust, at the expense of the recovery time and overall agility;

Spadroon An antique weapon

Basket Hilt Also an antique;

Albion Brescia Spadona The standard offering from Albion Armourers Museum line;

A&A type XVIIIa A sword made by Arms&Armor;

AT 1401, 1516, 1404 Three swords from Angus Trim. The 1401 has been modified with a lighter pommel than what it had originally.

In both cases the methodology used was slightly different from that described in this paper, though it gives exactly the same amount of information. The data is not the raw measured information but is computed from it. I have checked on my swords that both methods give the same results with a very acceptable precision. I hope that the data included in table 1 can provide a useful comparison point to anyone willing to measure his collection. For a more visual analysis, I also include a graph of the dynamic data on figure 9.

Name	Blade length (mm)	Hilt length (mm)	Dynamic length (mm)	Blade weight (g)	Cross weight (g)	Blade presence (%)
A&A Milanese	956	134	674	272	1074	20
Darkwood rapier	1068	142	791	156	942	14
Boken 1	749	266	589	179	393	31
Boken 2	730	285	554	279	514	35
Iaito	745	260	585	201	608	25
Ninja to	510	170	315	287	479	37
Albion Squire	820	160	606	219	888	20
AT Type XI	845	135	604	339	670	34
Purpleheart waster	975	245	630	309	398	44
Napoleonic briquet	590	130	423	231	1069	18
G.F. XV-1	745	150	650	151	939	14
G.F. XV-2	805	175	693	149	881	14
G.F. Spatha	815	135	580	320	542	37
G.F. XVI-1	824	158	710	194	994	16
G.F. XVI-2	785	160	676	209	951	18
A&A Montante	1150	400	1122	228	1956	10
Synthetic	885	180	577	222	542	29
A&A Cavalier	914	131	825	240	1250	16
Spadroon	800	165	692	192	1108	15
Basket Hilt	889	178	698	175	1020	15
Brescia Spadona	895	263	647	230	1304	15
AT1401	889	211	584	259	822	24
AT1516	838	267	686	143	1144	11
A&A type XVIIIa	800	231	552	211	1319	14
AT1404	724	165	495	261	1069	20

Table 1: Dynamic data gathered on swords by myself and Thom Ryan.

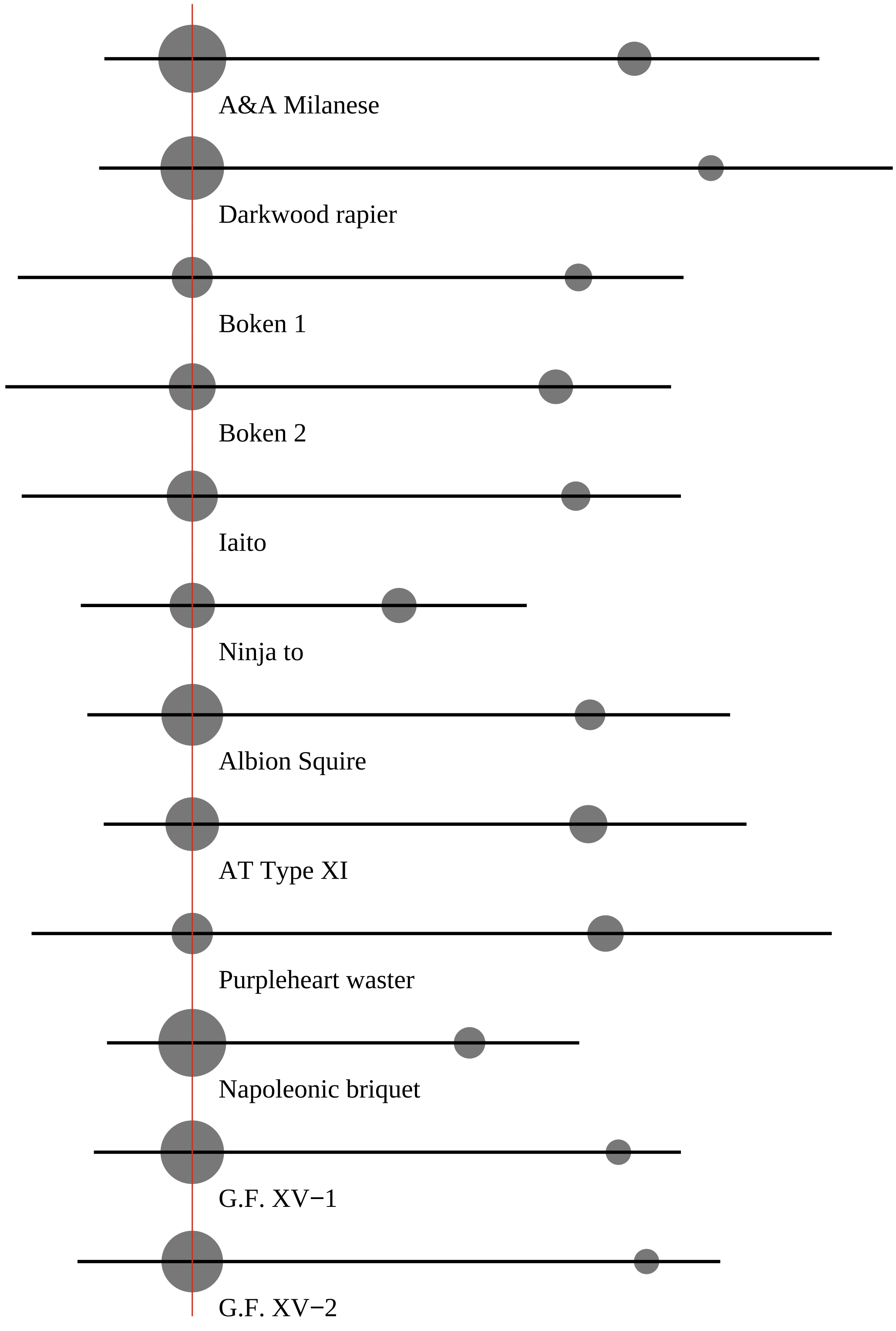

Figure 7: A visual representation of the two mass equivalent objects for all the measured weapons. The black line is the axis of the weapon. The red line intersects it at the junction cross-handle (all weapons were aligned according to this location in the figure). The area of the grey discs is proportional to the blade weight and the cross weight. The distance between them gives the dynamic length.

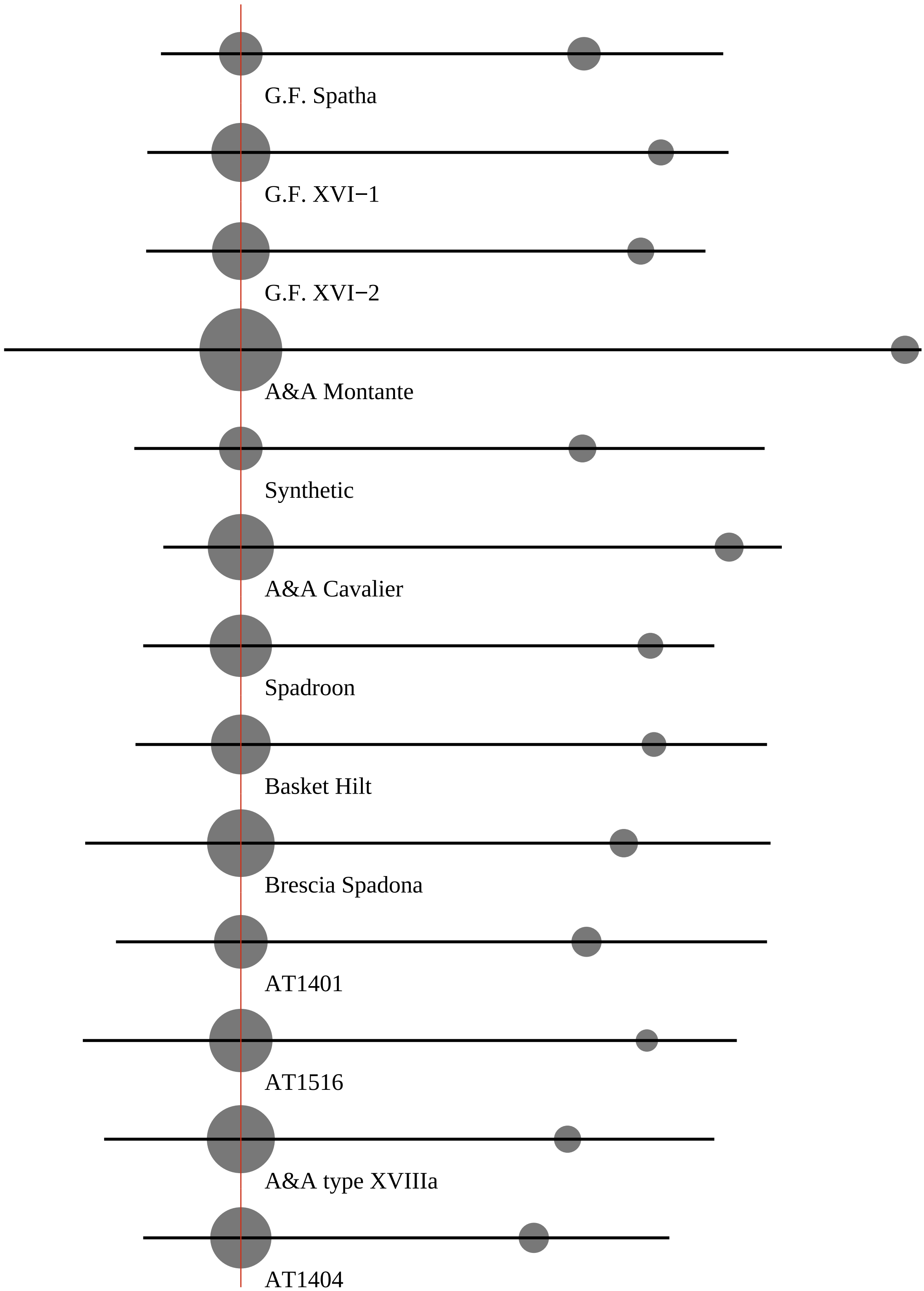

Figure 8: The rest of the data in the same format as on figure 7.

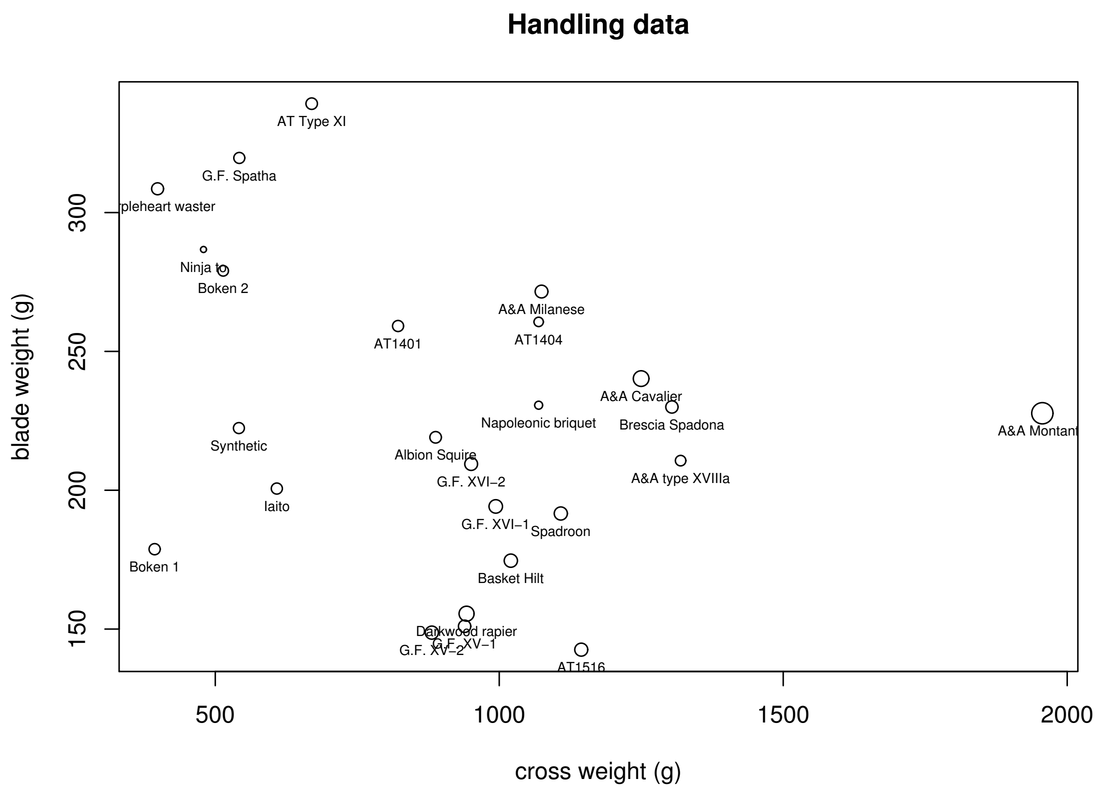

Figure 9: A plot summarizing the essential handling data measured on real swords. The x-axis is the cross weight, the y-axis is the blade weight, and the size of the circles is proportional to dynamic length.

B Mathematical demonstrations

In the following it is assumed that the reader is familiar with many mathematical operations, among which integration, differential equations, linear algebra and solving system of equations. These will not be explained again. I am unsure of how far in the educative system of each country around the world one would have to go to master these notions. I expect that anyone with an engineering background would understand the manipulations and definitions.

The demonstrations are of unequal difficulty. They are here because I feel it is important to have them in such a document, so that the interested reader does not have to rework everything by himself, and so that the assumptions and models are exposed to scrutiny in their details. They are also good examples of how to manipulate and think about the dynamics. However, as I said before, understanding these demonstrations is not required in order to use the results exposed in the core of the article.

I tried to rate the difficulty of the various sections by adding a number of ☕ next to the titles. The more coffees, the more difficult the part is. I do not drink coffee myself so the estimates could be a bit off...

B.1 Newton's equations and the definition of the mass distribution properties ☕

As per the assumptions made at the beginning, we will not give the most general definitions here, but rather simpler forms that pertain to one-dimensional objects moving in a two-dimensional plane. Points on the weapon have one coordinate along the axis of the weapon, that I denote by the lower-case of the name of the point. The coordinate value increases as you go towards the tip. The origin could be arbitrarily chosen, but setting things up so that the center of gravity G has coordinate 0 along the axis of the weapon simplifies the expressions.

At each point X along the axis of the weapon, we define $\mu(x)$, the linear density of the weapon at coordinate x. It is such that the mass of an infinitely thin section of the weapon comprised between positions x and $x + \mathrm{d}x$ is the product $\mu(x)\mathrm{d}x$. It could be thought off as virtually splitting the weapon in a great many tiny slices.

In the two-dimensional case, if you apply a force $\mathbf{F}$ and a torque C at point H, the speed of the weapon's center of gravity ($\mathbf{V}$) and its speed of rotation (Ω) vary according to:

$$\begin{aligned} M\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} &= \mathbf{F} \\ Mk^2\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= C + \mathbf{GH} \times \mathbf{F} \end{aligned} \tag{1}$$

Here $\frac{\mathrm{d}}{\mathrm{d}t}$ is used as the classical notation denoting variation in time, and $\times$ is the cross product of vectors. The definitions of the mass distribution properties (M, G, and k) are simply a matter of integration.

The mass of the weapon is:

$$M = \int_{weapon} \mu(x)\mathrm{d}x \tag{2}$$

The position of the center of gravity G is:

$$0 = g = \frac{1}{M}\int_{weapon} x\mu(x)\mathrm{d}x \tag{3}$$

(remember, G is chosen as the origin of the weapon's coordinate system, hence $g = 0$ by definition)

And finally, the radius of gyration around the center of gravity is:

$$k = \sqrt{\frac{1}{M}\int_{weapon} (x-g)^2\mu(x)\mathrm{d}x} \tag{4}$$

($g = 0$ here but is left in the definition for clarity, as it is possible to define radius of gyration around any point)

On statistical distributions, if we consider $\mu(x)$ as a probability density function, then $M = 1$ (it is a property of these functions), g is by definition the average, and k is the standard deviation.

B.2 Two-mass equivalence ☕☕

Let us have a weapon of mass M, center of gravity G, and radius of gyration k. As usual, we will take all measurements of lengths relative to G. Let us choose a point R at abscissa r. We are looking for the system of two masses, one m_1 at R, one m_2 at another point P, that has the same total mass, radius of gyration, and center of gravity as the weapon. That gives the following system, to be solved for p, m_1 and m_2:

$$\begin{cases} m_1 & + & m_2 & = & M \\ m_1\,r & + & m_2\,p & = & Mg = 0 \\ m_1\,r^2 & + & m_2\,p^2 & = & Mk^2 \end{cases} \tag{5}$$

With the hypothesis that $r \neq p$, we can solve the first two equations in m_1 and m_2 like a linear system:

$$\begin{bmatrix} 1 & 1 \\ r & p \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} M \\ 0 \end{bmatrix}$$

Assuming that $r \neq p$, the matrix on the left-hand side can easily be inverted, giving:

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \dfrac{p}{p-r} & -\dfrac{1}{p-r} \\ -\dfrac{r}{p-r} & \dfrac{1}{p-r} \end{bmatrix} \begin{bmatrix} M \\ 0 \end{bmatrix}$$

And finally:

$$m_1 = M\frac{p}{p-r}$$
$$m_2 = -M\frac{r}{p-r}$$

We then replace m_1 and m_2 with their respective values in the third equation of the system (5), finding:

$$\begin{aligned}
Mk^2 &= Mr^2\frac{p}{p-r} - Mp^2\frac{r}{p-r} \\
&= M\left(\frac{pr^2 - rp^2}{p-r}\right) \\
&= M\left(\frac{rp(r-p)}{p-r}\right) \\
&= -Mrp
\end{aligned}$$

To conclude, we end up with the following solution:

$$\begin{cases}
m_1 = M\dfrac{p}{p-r} \\
m_2 = M\dfrac{r}{r-p} \\
p = -\dfrac{k^2}{r}
\end{cases} \tag{6}$$

B.3 Pendulums and the 'waggle test' ☕☕☕

B.3.1 Description of the system

The fixed origin of the coordinates is O. The sword is supposed to hang from point R that can move horizontally but not vertically, and is always situated on the $\mathbf{x}$ axis. A force $\mathbf{F}$ is applied at this point, with vertical and horizontal components. The sword is also subject to the action of the gravity along the vertical, represented as a vertical force $\mathbf{P}$ at the sword's center of gravity. The angle between the axis of the sword and the vertical is θ.

To sum up, the system is parametrized like this:

$$\begin{aligned}
\mathbf{OR} &= x\ \mathbf{x} \\
\mathbf{RG} &= r\sin(\theta)\ \mathbf{x} - r\cos(\theta)\ \mathbf{y} \\
\mathbf{F} &= F_x\ \mathbf{x} + F_y\ \mathbf{y} \\
\mathbf{P} &= -Mg\ \mathbf{y}
\end{aligned} \tag{7}$$

On figure 10 the main variables are shown.

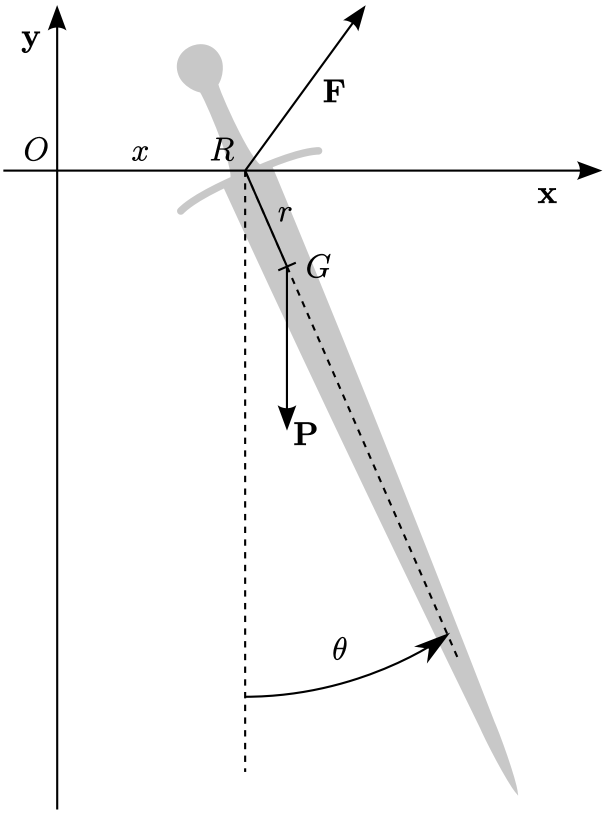

Figure 10: The model used to demonstrate the properties of swords as pendulums and the waggle test.

B.3.2 Generic equations of the system

We can write Newton's equation for the system like this:

$$\begin{cases} M\dfrac{\mathrm{d}^2\mathbf{OG}}{\mathrm{d}t^2} = \mathbf{F} + \mathbf{P} \\ Mk^2\,\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = (\mathbf{F} \times \mathbf{RG}) \cdot \mathbf{z} \end{cases} \tag{8}$$

Replacing **F**, **P** and **RG** by their expressions given earlier, we find:

$$\begin{cases} M\dfrac{\mathrm{d}^2\mathbf{OG}}{\mathrm{d}t^2} = F_x\ \mathbf{x} + (F_y - Mg)\ \mathbf{y} \\ Mk^2\,\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -F_y r\sin(\theta) - F_x r\cos(\theta) \end{cases} \tag{9}$$

The next step is the computation of the second derivative of **OG**. Noting first and second derivatives with $\dot{}$ and $\ddot{}$, we find:

$$\begin{aligned} \mathbf{OG} &= (x + r\sin(\theta))\ \mathbf{x} - r\cos(\theta)\ \mathbf{y} \\ \frac{\mathrm{d}\mathbf{OG}}{\mathrm{d}t} &= (\dot{x} + r\dot{\theta}\cos(\theta))\ \mathbf{x} + r\dot{\theta}\sin(\theta)\ \mathbf{y} \\ \frac{\mathrm{d}^2\mathbf{OG}}{\mathrm{d}t^2} &= (\ddot{x} + r\ddot{\theta}\cos(\theta) - r\dot{\theta}^2\sin(\theta))\ \mathbf{x} + (r\ddot{\theta}\sin(\theta) + r\dot{\theta}^2\cos(\theta))\ \mathbf{y} \end{aligned} \tag{10}$$

We introduce this result in 9, and project on **x** and **y**, finally finding the following system of 3 equations:

$$\begin{cases} M\ddot{x} + Mr\ddot{\theta}\cos(\theta) - Mr\dot{\theta}^2\sin(\theta) = F_x \\ Mr\ddot{\theta}\sin(\theta) + Mr\dot{\theta}^2\cos(\theta) = F_y - Mg \\ -F_y r\sin(\theta) - F_x r\cos(\theta) = Mk^2\ddot{\theta} \end{cases} \tag{11}$$

B.3.3 Expression of the system for small displacements

The system 11 is exact, but hard to solve. As is often done with the simple pendulum, we will rather solve an approximate system, valid for small θ. We make a first order approximation, which gives $\sin(\theta) \approx \theta$, $\cos(\theta) \approx 1$, $\ddot{\theta}\theta \approx 0$, and $\dot{\theta}^2 \approx 0$. The system takes a simpler form:

$$\begin{cases} M\ddot{x} + Mr\ddot{\theta} = F_x \\ 0 = F_y - Mg \\ -F_y r\theta - F_x r = Mk^2\ddot{\theta} \end{cases} \tag{12}$$

B.3.4 Pendular motion

Starting from this result, we are able to derive the equation driving the motion in the case of a fixed point R. We just have to assume x as a constant in time, and thus in particular $\ddot{x} = 0$. This leads, after replacing F_y by its expression, to the system:

$$\begin{cases} Mr\ddot{\theta} = F_x \\ -Mgr\theta - F_x r = Mk^2\ddot{\theta} \end{cases} \tag{13}$$

Eliminating F_x and regrouping the terms, we get:

$$\begin{aligned} -Mgr\theta - Mr^2\ddot{\theta} &= J\ddot{\theta} \\ -Mgr\theta &= M(k^2 + r^2)\ddot{\theta} \end{aligned} \tag{14}$$

In order to simplify further, we can use P, the center of oscillation associated to R. Its position relative to the center of gravity of the weapon is $p = \frac{k^2}{r}$, as shown in appendix B.2 (pay attention to the convention of sign for the distances, here p and r are both positive). The equation simplifies further:

$$\begin{aligned} -Mgr\theta &= (Mrp + Mr^2)\ddot{\theta} \\ g\theta &= -(p + r)\ddot{\theta} \end{aligned} \tag{15}$$

And finally, defining $l = p + r$ the length between the fixed point R and its center of oscillation P, we find:

$$g\theta = -l\ddot{\theta} \tag{16}$$

This is exactly the equation of a simple pendulum (a point mass attached to a string) of length l. It is a common calculation to find the period of oscillation, something well-known and demonstrated in any physics textbook. As a reminder the period is simply $T = 2\pi\sqrt{\frac{l}{g}}$. That formula can be used to measure the center of oscillation in an other way, by timing the oscillations of the sword.

B.3.5 Precision of the 'waggle test'

For the study of the 'waggle test', we are looking for the behaviour of the system during forced oscillations. That is, the tester imposes a periodic displacement of R on the $\mathbf{x}$ axis, and the object responds with a periodic variation of the angle θ. Using complex numbers for the notation, we can write this as $x = x_0 e^{j\omega t}$, $\theta = \theta_0 e^{j\omega t}$, where $\omega(\mathrm{s}^{-2})$ is the pulsation of the excitation.

This has the advantage of suppressing derivatives, because $\frac{\mathrm{d}e^{j\omega t}}{\mathrm{d}t} = j\omega e^{j\omega t}$. This leads to another version of the system written in 12:

$$\begin{cases} (-M\omega^2 x_0 - Mr\omega^2\theta_0)e^{j\omega t} = F_x = F_0 e^{j\omega t} \\ F_y = Mg \\ -Mgr\theta_0 - F_0 r = -Mk^2\omega^2\theta_0 \end{cases} \tag{17}$$

It appears that the periodicity of x and θ enforces the periodicity of F_x. We then find two different expressions for the amplitude F_0 of the force:

$$\begin{cases} F_0 = -M(\omega^2 x_0 + r\omega^2\theta_0) \\ F_0 = M\dfrac{k^2\omega^2 - gr}{r}\theta_0 \end{cases} \tag{18}$$

These equalities can then be used to find an expression of x_0 as a function of θ_0:

$$\begin{aligned} \omega^2 x_0 + r\omega^2\theta_0 &= \frac{-k^2\omega^2 + gr}{r}\theta_0 \\ x_0 &= \frac{-k^2\omega^2 + gr - r^2\omega^2}{r\omega^2}\theta_0 \end{aligned} \tag{19}$$

To simplify the expression for x_0, again we use P, the center of oscillation associated to R. The expression for x_0 becomes:

$$x_0 = \frac{g - (r+p)\omega^2}{\omega^2}\theta_0 \tag{20}$$

We will use this to find the precision of the 'waggle test'. During the test, we are looking for a point A of the weapon that stays fixed in space. This means that $\frac{\mathrm{d}\mathbf{OA}}{\mathrm{d}t} = \mathbf{0}$. The derivation of $\mathbf{OA}$ is very similar to that of $\mathbf{OG}$ in 10:

$$\begin{aligned} \mathbf{OA} &= \mathbf{OG} + a\sin(\theta)\mathbf{x} - a\cos(\theta)\mathbf{y} \\ \frac{\mathrm{d}\mathbf{OA}}{\mathrm{d}t} &= \left(\dot{x} + (r+a)\dot{\theta}\cos(\theta)\right)\mathbf{x} + (r+a)\dot{\theta}\sin(\theta)\ \mathbf{y} \end{aligned} \tag{21}$$

With the approximation we have made earlier, an expression of a becomes:

$$\begin{aligned} 0 &= j\omega x_0 + (r+a)j\omega\theta_0 \\ a &= -\frac{x_0}{\theta_0} - r \end{aligned} \tag{22}$$

Thanks to the previous expression of x_0, the value of a can easily be found:

$$\begin{aligned} a &= -\frac{g - (r+p)\omega^2}{\omega^2} - r \\ &= \frac{(r+p)\omega^2 - g - r\omega^2}{\omega^2} \\ &= p - \frac{g}{\omega^2} \end{aligned} \tag{23}$$

The interest of the 'waggle test' is that A is meant to be very close to P. Noting the difference (the error of the measure) as $\varepsilon = p - a$, and using the link from the pulsation to the period of the oscillations $\omega = \frac{2\pi}{T}$, we find finally:

$$\varepsilon = \frac{T^2 g}{4\pi^2} \tag{24}$$

Or if what is desired is a given precision for the test:

$$T = 2\pi\sqrt{\frac{\varepsilon}{g}} \tag{25}$$

It is interesting to note that the precision becomes very high for quite low T. For example, for $\varepsilon = 1$cm, we find $T \approx 0.2$s. In fact the precision of the test itself is easily superior to the precision of the measure of the fixed spot on the weapon.